

Combinatorics of Fiedler Pencils with Repetitions

Gianna M. Del Corso Federico Poloni

Department of Computer Science, University of Pisa

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Fiedler linearizations I

$$\begin{bmatrix} & I & & \\ & & I & \\ & & & I \\ A_0 & A_1 & A_2 & A_3 \end{bmatrix} - x \begin{bmatrix} I & & & \\ & I & & \\ & & I & \\ & & & -A_4 \end{bmatrix}$$

is a **linearization** (=same eigenvalues and multiplicities) of the matrix polynomial

$$A_4x^4 + A_3x^3 + A_2x^2 + A_1x + A_0 \in \mathbb{C}[x]^{n \times n}.$$

Fiedler linearizations I

$$\begin{bmatrix} / & & & & \\ & / & & & \\ & & / & & \\ & & & / & \\ / & & & & A_0 \end{bmatrix} - x \begin{bmatrix} / & & & & \\ & / & & & \\ & & / & & \\ & & & / & \\ & & & & -A_4 \end{bmatrix}$$

$$\begin{bmatrix} / & & & & \\ & / & & & \\ & & / & & \\ & & & / & \\ / & & & & A_3 \end{bmatrix} \begin{bmatrix} / & & & & \\ & / & & & \\ & & / & & \\ & & & / & \\ / & & & & A_2 \end{bmatrix} \begin{bmatrix} / & & & & \\ & / & & & \\ & & / & & \\ & & & / & \\ / & & & & A_1 \end{bmatrix} \begin{bmatrix} A_0 & & & & \\ & / & & & \\ & & / & & \\ & & & / & \\ & & & & / \end{bmatrix}$$

is a **linearization** (=same eigenvalues and multiplicities) of the matrix polynomial

$$A_4x^4 + A_3x^3 + A_2x^2 + A_1x + A_0 \in \mathbb{C}[x]^{n \times n}.$$

Fiedler linearizations II

... and so is any product of the same factors in different order [Fiedler, '03, Antoniou-Vologiannidis '04]

$$\begin{aligned}
 & \underbrace{\begin{bmatrix} & A_0 & & \\ / & A_1 & & \\ & & / & \\ / & A_2 & A_3 & \end{bmatrix}}_{\text{product of factors}} \cdot \begin{bmatrix} / & & & \\ & / & & \\ & & / & \\ & & & -A_4 \end{bmatrix} \\
 & \begin{bmatrix} A_0 & & & \\ & / & & \\ & & / & \\ & & & / \end{bmatrix} \begin{bmatrix} / & & & \\ & / & & \\ & & / & \\ & & & / A_3 \end{bmatrix} \begin{bmatrix} / & & & \\ / & A_1 & & \\ & & / & \\ & & & / \end{bmatrix} \begin{bmatrix} / & & & \\ & / & & \\ & & / & \\ & & & / A_2 \end{bmatrix} \begin{bmatrix} / & & & \\ & / & & \\ & & / & \\ & & & / \end{bmatrix}
 \end{aligned}$$

Fiedler linearizations III

... and there are more with factors of the same kind. [Vologiannidis-Antoniou '11]

$$\begin{bmatrix} & & I \\ & A_0 & A_1 \\ A_0 & A_1 & A_2 \\ & I & A_2 & A_3 \end{bmatrix} - x \begin{bmatrix} & & & \\ & & I & \\ A_0 & & & A_1 \\ & & & A_2 \\ & & & & -A_4 \end{bmatrix}$$

$$= \begin{bmatrix} I & & & \\ & I & & \\ & & I & \\ & & & I \end{bmatrix} \begin{bmatrix} I & & & \\ & I & & \\ & & I & \\ & & & I \end{bmatrix} \begin{bmatrix} I & & & \\ & I & & \\ & & I & \\ & & & I \end{bmatrix} \begin{bmatrix} A_0 & & & \\ & I & & \\ & & I & \\ & & & I \end{bmatrix} \begin{bmatrix} I & & & \\ & I & & \\ & & I & \\ & & & I \end{bmatrix} \begin{bmatrix} I & & & \\ & I & & \\ & & I & \\ & & & I \end{bmatrix} \begin{bmatrix} A_0 & & & \\ & I & & \\ & & I & \\ & & & I \end{bmatrix} \\ -x \begin{bmatrix} I & & & \\ & I & & \\ & & I & \\ & & & I \end{bmatrix} \begin{bmatrix} I & & & \\ & I & & \\ & & I & \\ & & & I \end{bmatrix} \begin{bmatrix} I & & & \\ & I & & \\ & & I & \\ & & & -A_4 \end{bmatrix} \begin{bmatrix} A_0 & & & \\ & I & & \\ & & I & \\ & & & I \end{bmatrix}.$$

Plan

Finding new linearizations and using them numerically is a very active research area.

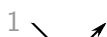
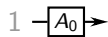
Special interest in finding structured (**symmetric**, **palindromic**...) linearizations of structured matrix polynomials.

[Mackey-Mackey-Mehl-Mehrmann '06]

In this talk: we study Fiedler linearizations using computational diagrams.

$$F_2 = \begin{bmatrix} I & & & \\ & I & & \\ & & A_2 & \\ & & & I \end{bmatrix} = \begin{array}{c} 1 \text{ ———} \\ 2 \begin{array}{c} \nearrow \\ \searrow \end{array} \\ 3 \begin{array}{c} \boxed{A_2} \\ \rightarrow \end{array} \\ 4 \text{ ———} \end{array}$$

Diagrams for elementary factors



$$\begin{bmatrix} A_0 & & & \\ & I & & \\ & & I & \\ & & & I \end{bmatrix}$$

F_0

$$\begin{bmatrix} & I & & \\ I & A_1 & & \\ & & I & \\ & & & I \end{bmatrix}$$

F_1

$$\begin{bmatrix} I & & & \\ & I & & \\ & & A_2 & \\ & & & I \end{bmatrix}$$

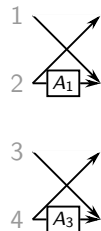
F_2

$$\begin{bmatrix} I & & & \\ & I & & \\ & & I & \\ & & & A_3 \end{bmatrix}$$

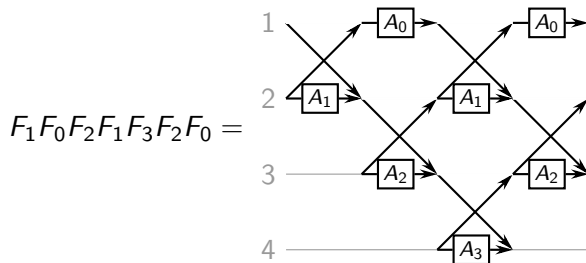
F_3

Commuting factors

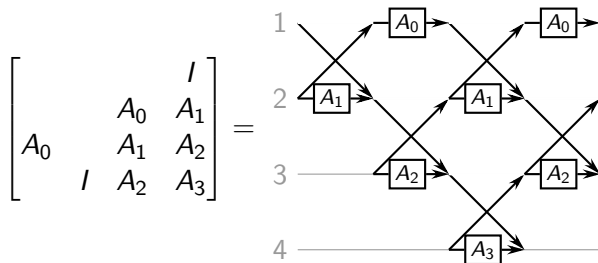
$$F_3 F_1 = F_1 F_3$$

$$\begin{bmatrix} I & & & \\ & I & & \\ & & I & \\ & & & A_3 \end{bmatrix} \begin{bmatrix} I & & & \\ & I & & \\ & & I & \\ & & & A_1 \end{bmatrix} = \begin{bmatrix} I & & & \\ & I & & \\ & & I & \\ & & & A_1 \end{bmatrix} \begin{bmatrix} I & & & \\ & I & & \\ & & I & \\ & & & A_3 \end{bmatrix} =$$


More complex diagrams

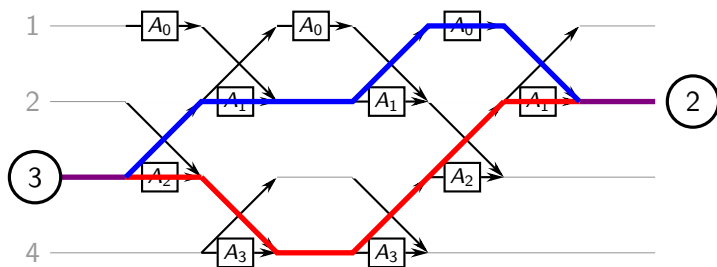


More complex diagrams



Block-Path correspondence

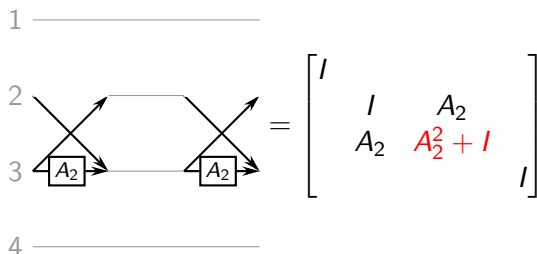
$$\begin{bmatrix} 0 & A_0^2 & A_0A_1 & 0 \\ I & A_1 & A_2 & A_3 \\ A_2 & A_1A_0 + A_2A_1 & A_1^2 + A_2^2 + A_0 & A_2A_3 \\ A_3 & A_3A_1 & A_3A_2 & A_3^2 + I \end{bmatrix}$$



Operation-freeness

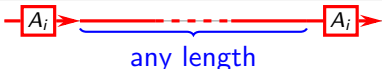
Desirable property of our linearizations: **no operations needed**.

Does not hold for all Fiedler products:



Theorem [Vologiannidis-Antoniou '11]

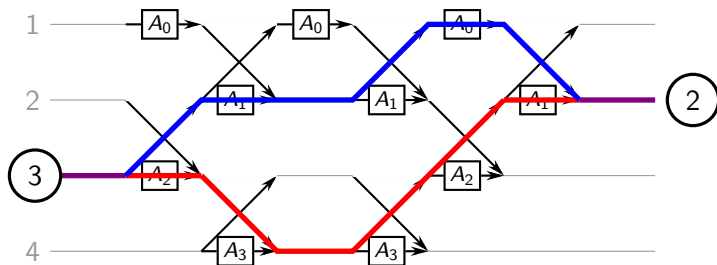
A Fiedler product is operation-free **unless** there are two factors F_k without a F_{k+1} in-between.

'Forbidden shape': 

Sketch of a (diagram-based) proof

Nontrivial part: take a non-operation-free product:

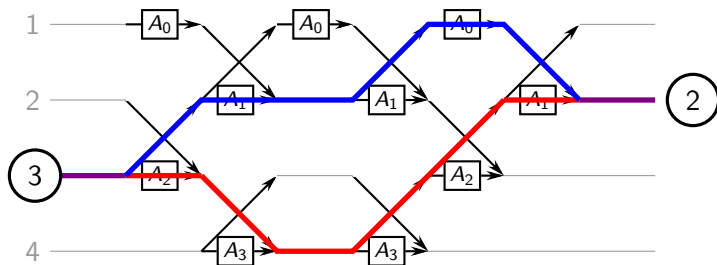
$$\begin{bmatrix} 0 & A_0^2 & A_0A_1 & 0 \\ I & A_1 & A_2 & A_3 \\ A_2 & A_1A_0 + A_2A_1 & A_1^2 + A_2^2 + A_0 & A_2A_3 \\ A_3 & A_3A_1 & A_3A_2 & A_3^2 + I \end{bmatrix}$$



Sketch of a (diagram-based) proof

Nontrivial part: take a non-operation-free product:

$$\begin{bmatrix} 0 & A_0^2 & A_0A_1 & 0 \\ I & A_1 & A_2 & A_3 \\ A_2 & A_1A_0 + A_2A_1 & A_1^2 + A_2^2 + A_0 & A_2A_3 \\ A_3 & A_3A_1 & A_3A_2 & A_3^2 + I \end{bmatrix}$$

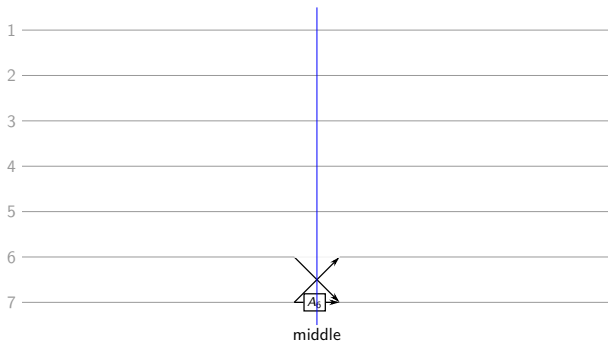


Forbidden shape at bottom of the lower path

The Middle Standard Form

Rule: draw each block as close to the middle as possible.

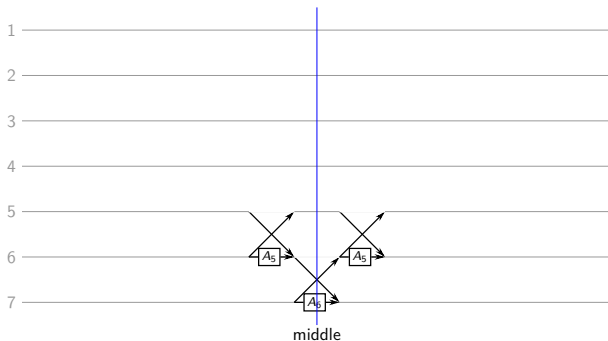
$F_4 F_5 F_1 F_2 F_3 F_0 F_1 F_4 F_6 F_3 F_5 F_4 F_3 F_2 F_1 F_0$



The Middle Standard Form

Rule: draw each block as close to the middle as possible.

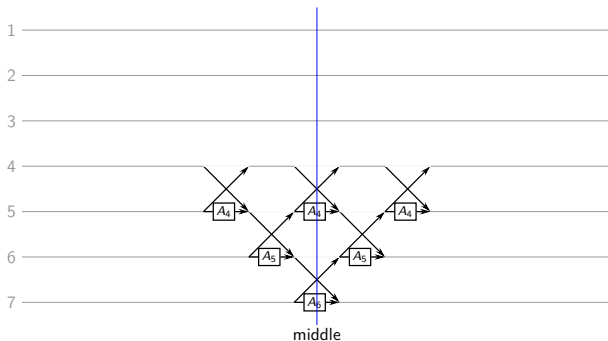
$F_4 F_5 F_1 F_2 F_3 F_0 F_1 F_4 F_6 F_3 F_5 F_4 F_3 F_2 F_1 F_0$



The Middle Standard Form

Rule: draw each block as close to the middle as possible.

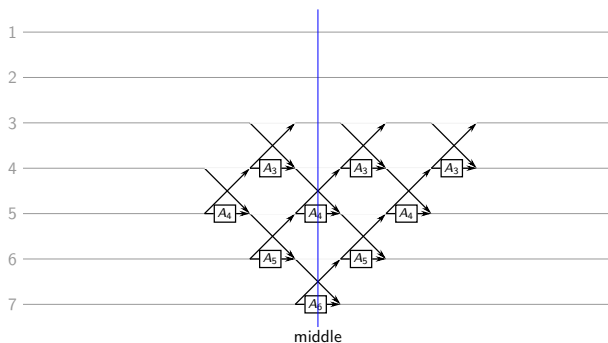
$F_4 F_5 F_1 F_2 F_3 F_0 F_1 F_4 F_6 F_3 F_5 F_4 F_3 F_2 F_1 F_0$



The Middle Standard Form

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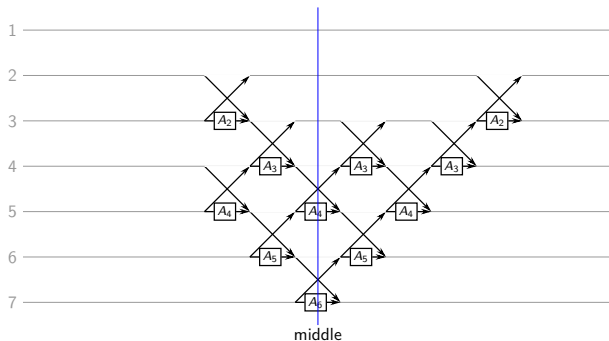
$F_4 F_5 F_1 F_2 F_3 F_0 F_1 F_4 F_6 F_3 F_5 F_4 F_3 F_2 F_1 F_0$



The Middle Standard Form

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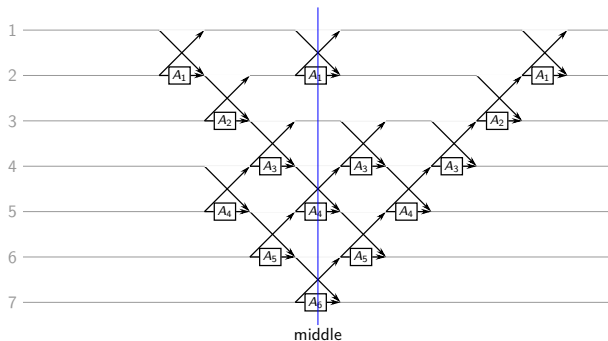
$$F_4 F_5 F_1 F_2 F_3 F_0 F_1 F_4 F_6 F_3 F_5 F_4 F_3 F_2 F_1 F_0$$



The Middle Standard Form

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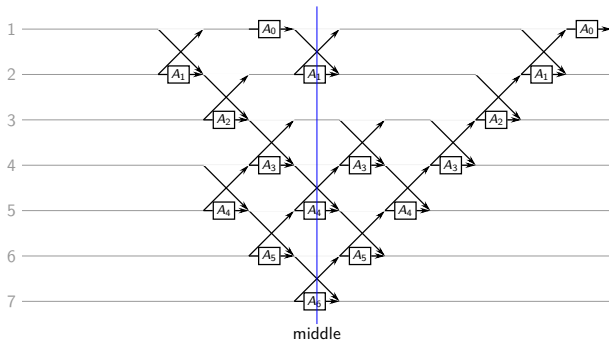
$$F_4 F_5 F_1 F_2 F_3 F_0 F_1 F_4 F_6 F_3 F_5 F_4 F_3 F_2 F_1 F_0$$



The Middle Standard Form

Rule: draw each block as close to the middle as possible.

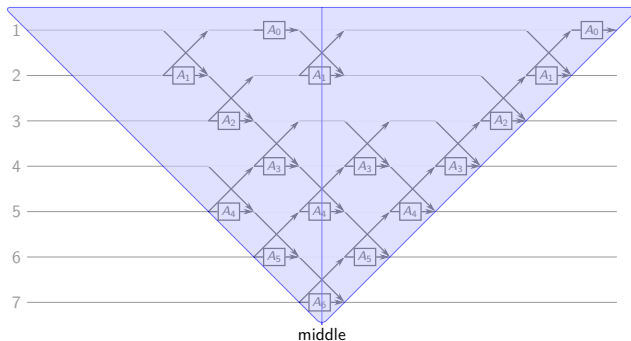
$$F_4 F_5 F_1 F_2 F_3 F_0 F_1 F_4 F_6 F_3 F_5 F_4 F_3 F_2 F_1 F_0$$



The Middle Standard Form

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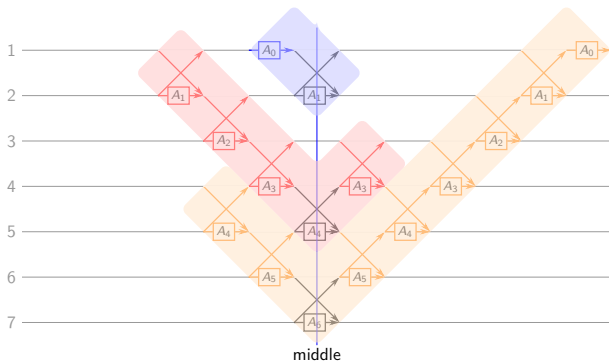
$$F_4 F_5 F_1 F_2 F_3 F_0 F_1 F_4 F_6 F_3 F_5 F_4 F_3 F_2 F_1 F_0$$



The Middle Standard Form

Rule: draw each block as close to the middle as possible.

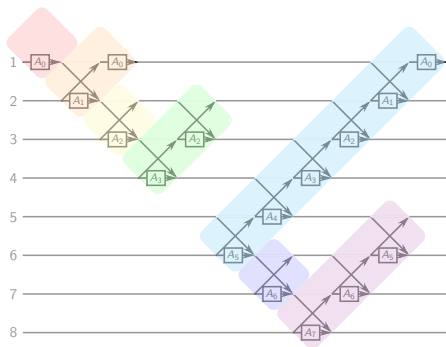
$$F_4 F_5 F_1 F_2 F_3 F_0 F_1 F_4 F_6 F_3 F_5 F_4 F_3 F_2 F_1 F_0$$



$$F = \underbrace{F_4 F_5 F_1 F_2 F_3}_{S_3} \underbrace{F_0}_{S_2} \underbrace{F_1 F_4 F_6}_{S_1} \underbrace{I}_M \underbrace{F_3}_{T_1} \underbrace{F_5 F_4 F_3 F_2 F_1 F_0}_{T_2 T_3}$$

Other standard forms

Row standard form: blocks as far left as possible (inside a triangular frame)



Naturally arranged in diagonals.

Column standard form: blocks as far right as possible.

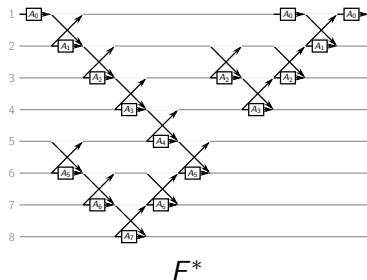
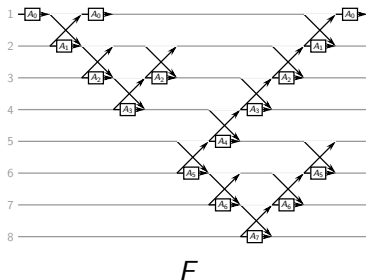
Corollary [Vologiannidis-Antoniou '11]

For a degree- d $A(x)$, there are $(d + 1)!$ operation-free products.

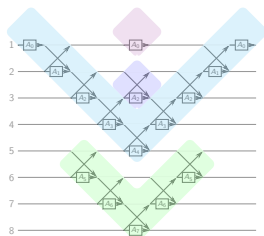
Symmetric matrix polynomials

If $A(x)$ is **symmetric** (i.e., $A_i = A_i^*$ for all i), then

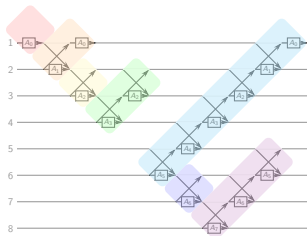
- the blocks $F_i = F_i^*$ are symmetric;
- a diagram for F^* is obtained by **flipping horizontally** one for F .



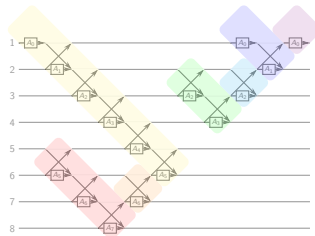
The MSF reveals symmetry



Middle standard form



Row standard form



Column standard form

Symmetric Fiedler products

Theorem

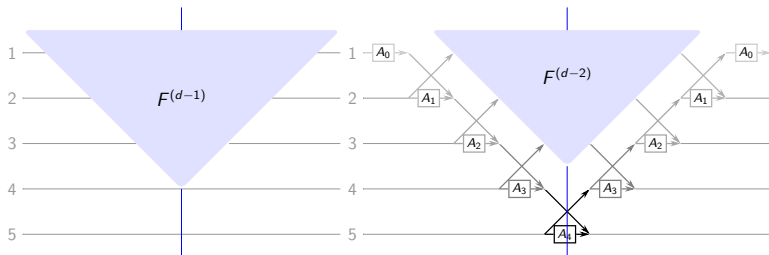
$F = F^* \iff$ The MSF of F has left-right symmetry.

We can count symmetric Fiedler products through their MSF:

Theorem

Let Σ_d be the number of different operation-free symmetric Fiedler products constructed from **symmetric** $A(x)$ of degree d . Then,

$$\Sigma_d = \Sigma_{d-1} + d \Sigma_{d-2}, \quad \Sigma_1 = 2, \Sigma_2 = 4.$$



Looking up this sequence

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founded in 1964 by N. J. A. Sloane

2, 4, 10, 26, 76, 232, 764, 2620, 9496, 35696, 140152, 568504, 2390480

Search

[Hints](#)

(Greetings from [The On-Line Encyclopedia of Integer Sequences!](#))

Search: **seq:2,4,10,26,76,232,764,2620,9496,35696,140152,568504,2390480**

Displaying 1-1 of 1 result found.

page 1

Sort: relevance | [references](#) | [number](#) | [modified](#) | [created](#) Format: long | [short](#) | [data](#)

[A000085](#) Number of self-inverse permutations on n letters, also known as involutions; number of Young tableaux with n cells. +20
263
(Formerly M1221 N0469)

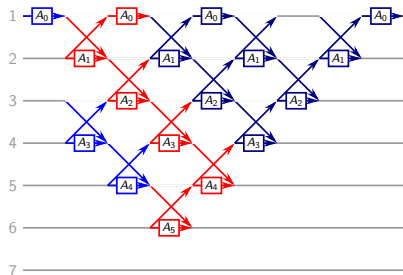
1, 1, **2**, **4**, **10**, **26**, **76**, **232**, **764**, **2620**, **9496**, **35696**, **140152**, **568504**, **2390480**, 10349536, 46206736, 211799312, 997313824, 4809701440, 23758664096, 119952692896, 618884638912, 3257843882624, 17492190577600, 95680443760576, 532985208200576, 3020676745975552 ([list](#); [graph](#); [refs](#); [listen](#); [history](#); [text](#); [internal format](#))

OFFSET 0,3

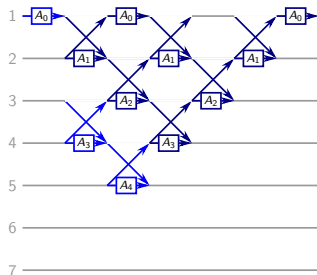
COMMENTS Getu (1991) says these numbers are also known as "telephone numbers". - [N. J. A. Sloane](#), Nov 23 2015
a(n) is also the number of n X n symmetric permutation matrices.
a(n) is also the number of matchings (Hosoya index) in the complete graph K(n).
- Ola Veshtha (olaveshta(AT)my-deja.com), Mar 25 2001.
Equivalently, this is the number of graphs on n labeled nodes with degrees at most 1. - [Don Knuth](#), Mar 31 2008
a(n) is also the sum of the degrees of the irreducible representations of the symmetric group S_n - Avi Peretz (nj(AT)netvision.net.il), Apr 01 2001
a(n) is the number of partitions of a set of n distinguishable elements into sets of size 1 and 2. [Karel A. Benson](#), Apr 22 2003

Infix pairs

Pairs of op-free products with and without an infix 'zig-zag' path.



LZR

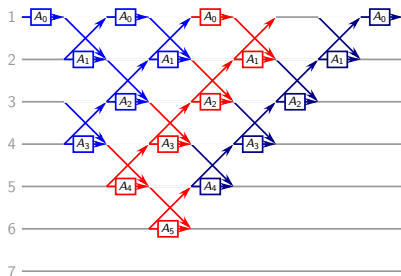


LR

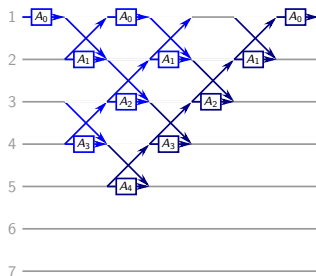
Basic building block for Fiedler-type linearizations.

Infix pairs

Pairs of op-free products with and without an infix 'zig-zag' path.



LZR

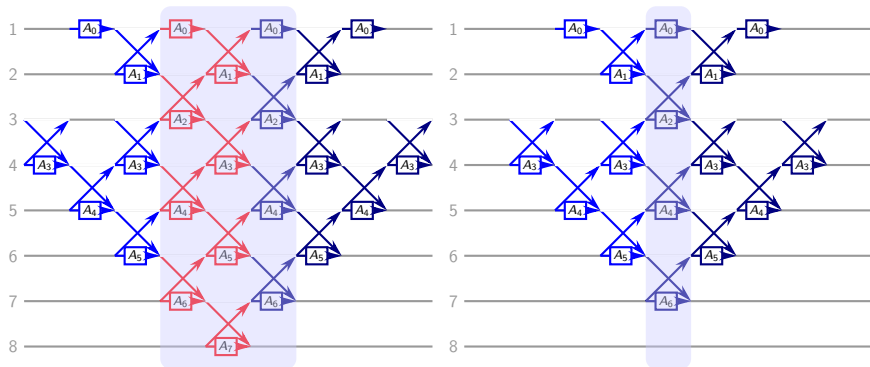


LR

Basic building block for Fiedler-type linearizations.

Warning: coloring not uniquely determined.

Standard form for symmetric infix pairs



Theorem

In a symmetric infix pair (up to recoloring / equivalence): the 3 middle columns are completely filled and contain entirely the zig-zag path.

Similar form in [Bueno-Dopico-Furtado-Rychnovsky '15].

Counting infix pairs

Theorem

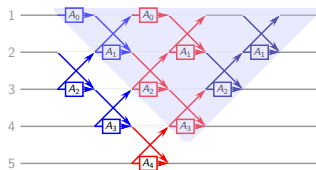
The number of **symmetric** op-free infix pairs (for $A(x)$ of degree d) is

$$(d-1)!! = (d-1)(d-3)(d-5)\cdots(2 \text{ or } 1).$$

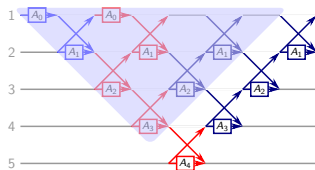
Theorem

The number Π_d of op-free infix pairs (for $A(x)$ of degree d) satisfies

$$\Pi_d = 2d\Pi_{d-1} - (d-1)^2\Pi_{d-2}, \quad \Pi_1 = 1, \Pi_2 = 3.$$



Left extension



Right extension

Number of infix pairs

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1, 3, 14, 85, 626, 5387, 52882, 582149, 7094234, 94730611, 1374650042

Search

[Hints](#)

(Greetings from [The On-Line Encyclopedia of Integer Sequences!](#))

Search: **seq:1,3,14,85,626,5387,52882,582149,7094234,94730611,1374650042**

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page 1

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A005189 Number of n-term 2-sided generalized Fibonacci sequences.
(Formerly M2976)

+20
1

1, 1, **1, 3, 14, 85, 626, 5387, 52882, 582149, 7094234, 94730611, 1374650042**, 21529197077, 361809517954, 6492232196699, 123852300381986, 2502521367966277, 53379537613065002, 1198434678728086019, 28245547605034208074, 697186985180529270101 ([list](#); [graph](#); [refs](#); [listen](#); [history](#); [text](#); [internal format](#))

OFFSET 0,4

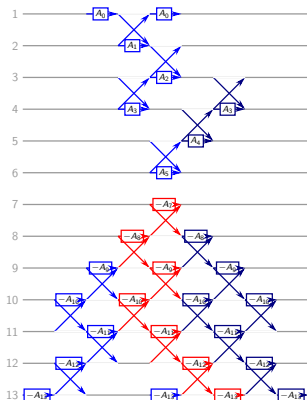
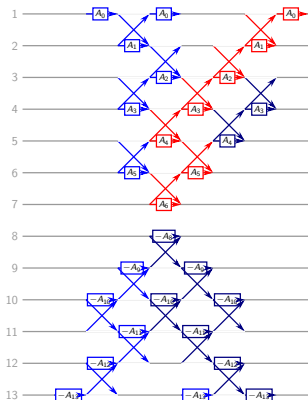
REFERENCES N. J. A. Sloane and Simon Plouffe, The Encyclopedia of Integer Sequences, Academic Press, 1995 (includes this sequence).

LINKS N. J. A. Sloane, [Table of n, a\(n\) for n = 0..250](#)
C. Banderier, H.-K. Hwang, V. Ravelomanana and V. Zacharovas, [Analysis of an exhaustive search algorithm in random graphs and the n^{c log n}-asymptotics](#), SIAM J. Discrete Math., 28(1), 342-371. (30 pages), DOI:10.1137/130916357. - From [N. J. A. Sloane](#), Dec 23 2012
Peter C. Fishburn, Peter C. Marcus-Roberts, Fred S. Roberts, [Unique finite difference measurement](#), SIAM J. Discrete Math. 1 (1988), no. 3, 334-354.
P. C. Fishburn, A. M. Odlyzko and F. S. Roberts, [2-sided generalized Fibonacci sequences](#), Fib. Quart., 27 (1989), 352-361.

FORMULA If $n \leq 2$ then $a(n) = 1$ otherwise $a(n) = 2^{(n-1)}a(n-1) - (n-2)^2a(n-2)$.
E.g.f.: $(e^*Ei(1/(x-1)) - e^*Ei(-1-1)/(e^{(x/(x-1))}*(x-1)))$, where Ei is the

Finally, linearizations

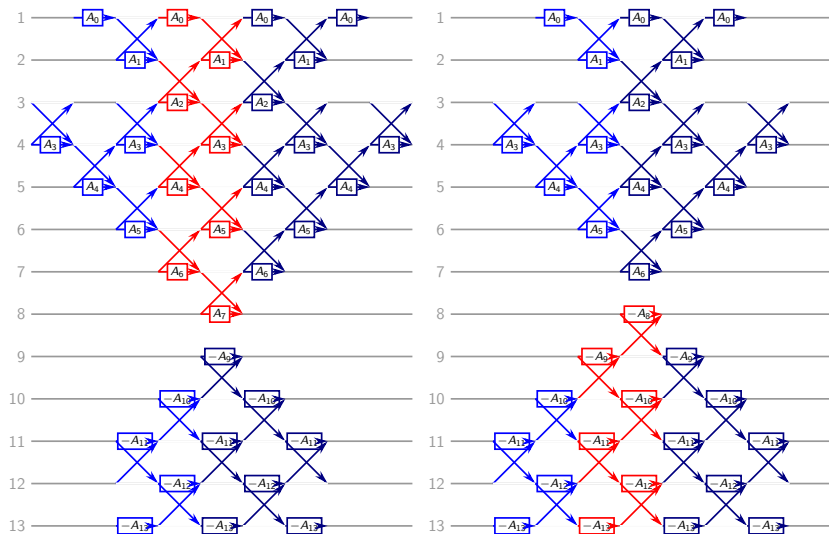
Fiedler pencil with repetitions = infix pair on the top + an analogous pair of inverse Fiedler matrices



$$L(Z^+ - xZ^-)R.$$

Symmetric linearizations

Symmetric top pair + symmetric bottom pair.



Counting linearizations

Theorem

For a degree- d matrix polynomial $A(x)$, there are

$$\sum_{h=1}^d \Pi_h \Pi_{d+1-h}$$

different Fiedler pencils with repetitions.

Theorem

For a degree- d **symmetric** matrix polynomial $A(x)$, there are

$$\sum_{h=1}^d (h-1)!!(d-h)!!$$

different **symmetric** Fiedler pencils with repetitions.

THE ON-LINE ENCYCLOPEDIA OF INTEGER SEQUENCES®

founded in 1964 by N. J. A. Sloane

[Hints](#)

(Greetings from [The On-Line Encyclopedia of Integer Sequences!](#))

Search: **seq:1,6,37,254,1958,16910**

Sorry, but the terms do not match anything in the table.

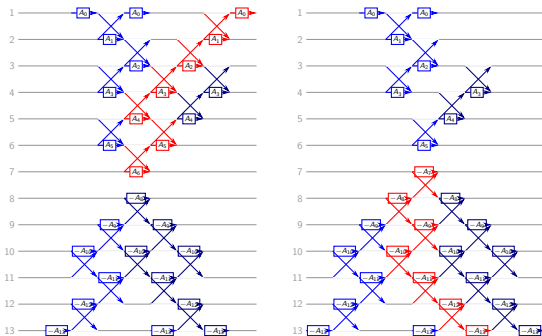
If your sequence is of general interest, please submit it using the [form provided](#) and it will (probably) be added to the OEIS! Include a brief description and if possible enough terms to fill 3 lines on the screen. We need at least 4 terms.

(Anti-)palindromic pencils

$A(x)$ **antipalindromic** matrix polynomial, i.e., $A_i = -A_{d-i}^*$.

Palindromic linearizations by making the substitution $x \rightarrow -x$

Are there antipalindromic Fiedler pencils with repetitions, i.e., $F - F^*x$?



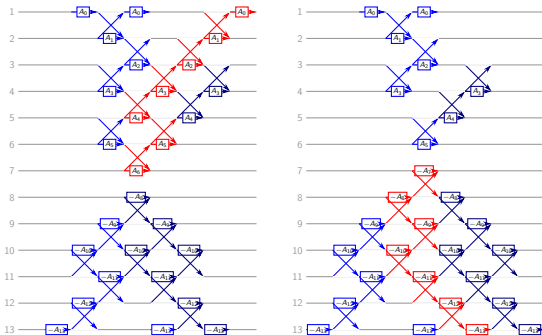
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No (apart from the trivial case $d = 1$).



J -antipalindromic pencils

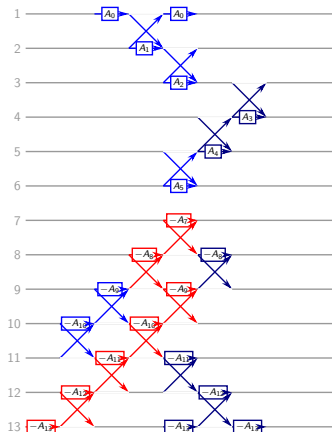
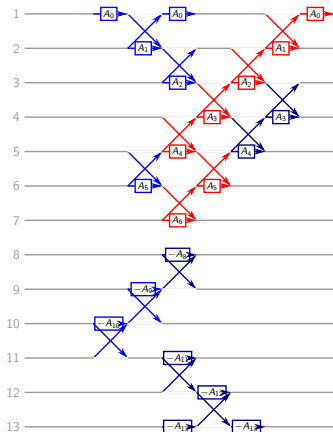
However, for **odd** d , one can have $J(F - F^*x)$ antipalindromic [Bueno-Furtado '12], with

$$J = \begin{bmatrix} & & & I_n \\ & & & \\ & & I_n & \\ & & \ddots & \\ I_n & & & \end{bmatrix}.$$

Remark: for **even** d , there are **no general constructions** for palindromic linearizations (not only Fiedler ones). [Mackey-Mackey-Mehl-Mehrmann '11]

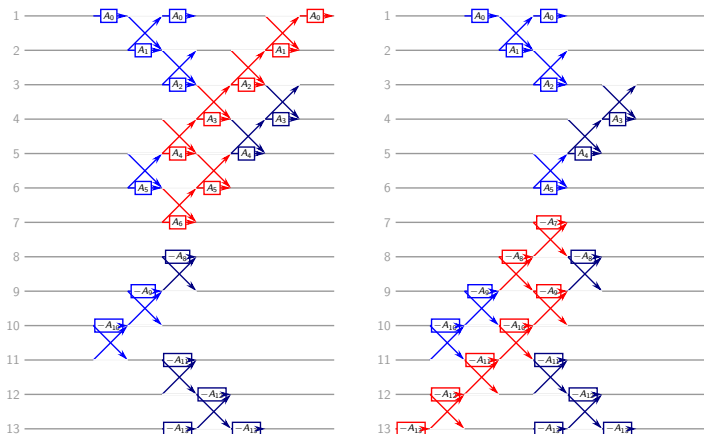
J -antipalindromic Fiedler pencils

$J = \text{up/down flip}$. A pencil $F - xG$ is J -antipalindromic if G can be obtained by F with an **up-down** and a **left-right** flip. E.g.:



J -antipalindromic Fiedler pencils

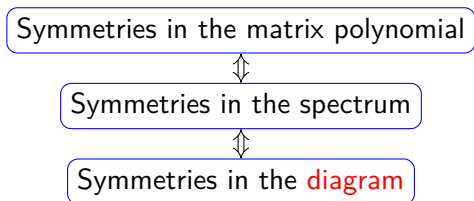
$J = \text{up/down flip}$. A pencil $F - xG$ is J -antipalindromic if G can be obtained by F with an **up-down** and a **left-right** flip. E.g.:



There are $\frac{\prod_{d+1}}{2}$ of them.

Conclusions

- Diagrams are a powerful tool to understand, characterize and **count** Fiedler-type pencils.
- New proofs and results, standard and special forms.
- Some issues ignored in this talk, e.g., different diagrams \leftrightarrow different matrices?



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Symmetries in the matrix polynomial



Symmetries in the spectrum



Symmetries in the **diagram**

Thanks for your attention!